Electromagnetic waves in NUT space: Solutions to the Maxwell equations

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Abstract

In this paper, using the Newman-Penrose formalism, we find the Maxwell equations in NUT space and after separation into angular and radial components solve them analytically. All the angular equations are solved in terms of Jaccobi polynomials. The radial equations are transformed into Hypergeometric and Heun's equations with the right hand sides including terms of different order in the frequency of the perturbation which allow solutions in the expansion of this parameter.

PACS No. 04.20.-q

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I. INTRODUCTION

An effective way to understand and analyse characteristics of an spacetime is to study its behaviour under different kind of perturbations including the most studied one *electromagnetic perturbations*. Indeed interest in the analytic solutions of Maxwell's equations in curved spacetimes, specially in the well known solutions of Einstein field equations, arose from the study of stability of those spacetimes under electromagnetic perturbations. On the other hand the (Taub)-NUT solution of Einstein vacuum field equations [1,2] has found importance both in the discussions on magnetic monopoles in gauge theories [3-4] and in string theory [5-6], though by itself interpreted as the metric of a mass M endowed with a gravitomagnetic monopole with strength l, the so called NUT parameter or *magnetic mass* [7]. Therefore it seems appropriate to study electromagnetic waves in NUT space through the solutions of the Maxwell equations in this spacetime and use the results to obtain a deeper understanding of the physical aspects of NUT space .

In a recent paper [8] perturbations of the Kerr-Taub-NUT spacetime by massless fields of spin $s \leq 2$ are studied by introducing a master equation describing these fields. The authors have also discussed the coupling between the gravitomagnetic monopole moment l and the spin of the perturbation field in the context of gravitomagnetism. In [9] Klein-Gordon and Dirac equations in Kerr-NUT space have been studied with an emphasis on a duality transformation involving the exchange of mass and NUT parameters. It is shown that the same duality transformation also exchanges the scalar and spinor perturbations. In the present article, as stated in the abstract, we have a more restricted task in front of us and that is to study electromagnetic perturbations (massless spin 1 field) in NUT space (and not Taub-NUT space) but we intend here to solve the resulted equations more or less anlytically.

To do so we start with the fact that as in the case of the Kerr metric, because NUT space is stationary and axisymmetric [1], one could express a general solution to the Maxwell

[1]For a review on the physical symmetries of NUT space refer to [7] and [10].

equations in this space as a superposition of different modes with a time- and ϕ-dependence given by [11]

$$e^{i(\omega t+m\phi)} \tag{1}$$

where $m = 0, \pm 1, \pm 2, ...$ and ω is the frequency of each mode. Treating the electromagnetic field as a perturbation, it is natural to think of ω as a small parameter in terms of which the appropriate functions could be expanded if necessary.

II. MAXWELL EQUATIONS IN NUT SPACE

In this section using Newman-Penrose formalism [12] we separate the remaining two variables, r and θ and obtain the radial and angular equations involved. To do so we start with the NUT metric given by the line element

$$ds^2 = f(r)\,(dt - 2l\cos\theta d\phi)^2 - \frac{1}{f(r)}dr^2 - (r^2+l^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

where $f(r) = 1 - \frac{2(Mr+l^2)}{r^2+l^2} > 0$ and l is called the *magnetic mass* or NUT factor. Using the tangent vectors to the null geodesics in NUT space, one can easily show that the following set of null vectors form a null-tetrad basis for the above metric

$$\begin{aligned}
l^\mu &= (\frac{1}{f(r)}, 1, 0, 0) \quad , \quad l_\mu = (1, -\frac{1}{f(r)}, 0, -2l\cos\theta) \\
n^\mu &= (\frac{1}{2}, -\frac{f(r)}{2}, 0, 0) \quad , \quad n_\mu = (\frac{f(r)}{2}, \frac{1}{2}, 0, -l\cos\theta f(r)) \\
m^\mu &= \frac{1}{\sqrt{2(r^2+l^2)}}(2il\cot\theta, 0, 1, \mathrm{i}\csc\theta) \quad , \quad m_\mu = \sqrt{\frac{r^2+l^2}{2}}(0, 0, -1, -\mathrm{i}\sin\theta) \\
\bar{m}^\mu &= \frac{1}{\sqrt{2(r^2+l^2)}}(-2il\cot\theta, 0, 1, -\mathrm{i}\csc\theta) \quad , \quad \bar{m}_\mu = \sqrt{\frac{r^2+l^2}{2}}(0, 0, -1, \mathrm{i}\sin\theta)
\end{aligned} \tag{3}$$

Using the above null tetrad one can find the following spin coefficients of NUT metric;

$$\kappa = 0, \;\; \sigma = 0, \;\; \lambda = 0, \;\; \nu = 0, \;\; \tau = 0, \;\; \pi = 0$$

$$\rho = \frac{-r+\mathrm{i}l}{r^2+l^2} \quad ; \quad \mu = \frac{f(r)}{2}(\frac{-r+\mathrm{i}l}{r^2+l^2})$$

$$\epsilon = \frac{\mathrm{i}l}{2(r^2+l^2)} \quad ; \quad \gamma = \frac{1}{4}\left(f'(r) + \frac{\mathrm{i}lf(r)}{r^2+l^2}\right)$$
$$\alpha = \frac{-1}{2}\frac{\cot\theta}{\sqrt{2(r^2+l^2)}} \quad ; \quad \beta = \frac{1}{2}\frac{\cot\theta}{\sqrt{2(r^2+l^2)}} \tag{4}$$

Having found the spin coefficients of NUT metric we can now write the Maxwell equations in this space using the Newman-Penrose formalism [12]. For this we replace the antisymmetric Maxwell tensor $F_{\mu\nu}$ with the following three *complex* scalars [11];

$$\phi_0 = F_{13} = F_{\mu\nu}l^\mu m^\nu$$
$$\phi_1 = \frac{1}{2}(F_{12}+F_{43}) = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu)$$
$$\phi_2 = F_{42} = F_{\mu\nu}\bar{m}^\mu n^\nu \tag{5}$$

where F_{ij} $(i,j = 1,2,3,4)$ and $F_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ are the components of the Maxwell tensor in the tetrad and tensor bases respectively. One can invert equations (5) to find

$$F_{\mu\nu} = 2\left[\phi_0\bar{m}_{[\mu}n_{\nu]} + \phi_1(n_{[\mu}l_{\nu]} + m_{[\mu}\bar{m}_{\nu]}) + \phi_2 l_{[\mu}m_{\nu]}\right] + C\ .\ C$$

In terms of the above scalars the Maxwell equations become;

$$D\phi_1 - \delta^*\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2$$
$$D\phi_2 - \delta^*\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2$$
$$\delta\phi_1 - \mathcal{P}\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2$$
$$\delta\phi_2 - \mathcal{P}\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 \tag{6}$$

Where in the above equations $D, \mathcal{P}, \delta$ and δ^* are the special symbols for the basis vectors $\mathbf{l}, \mathbf{n}, \mathbf{m}$ and $\bar{\mathbf{m}}$ when they are considered as the directional derivatives. For NUT space these are;

$$\mathbf{l} \equiv D = \frac{1}{f(r)}\frac{\partial}{\partial t} + \frac{\partial}{\partial r}$$
$$\mathbf{n} \equiv \mathcal{P} = \frac{1}{2}\frac{\partial}{\partial t} - \frac{f}{2}\frac{\partial}{\partial r}$$

$$\mathbf{m} \equiv \delta = \frac{1}{\sqrt{2(r^2+l^2)}}(2\mathrm{i}l\cot\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial \phi})$$

$$\bar{\mathbf{m}} \equiv \delta^* = \frac{1}{\sqrt{2(r^2+l^2)}}(-2\mathrm{i}l\cot\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial \phi}) \tag{7}$$

Now if we use the general form (1) for the (t,ϕ)- dependence of the wave, the above operators take the following forms;

$$D = \partial_r + \mathrm{i}\frac{\omega}{f(r)}$$

$$\mathcal{P} = \frac{\mathrm{i}\omega}{2} - \frac{f(r)}{2}\partial_r = -\frac{f(r)}{2}D^\dagger$$

$$\delta = \frac{1}{\sqrt{2(r^2+l^2)}}(-2l\omega\cot\theta + \partial_\theta - m\csc\theta)$$

$$\delta^* = \frac{1}{\sqrt{2(r^2+l^2)}}(2l\omega\cot\theta + \partial_\theta + m\csc\theta) \tag{8}$$

Now we define the above operators in the following way;

$$\mathbf{l} = D = \mathcal{D}_0 \qquad , \qquad \mathbf{n} = \mathcal{P} = -\frac{f(r)}{2}\mathcal{D}_0^\dagger$$

$$\mathbf{m} = \delta = \frac{1}{\sqrt{2(r^2+l^2)}}\mathcal{L}_0^\dagger \qquad , \qquad \bar{\mathbf{m}} = \delta^* = \frac{1}{\sqrt{2(r^2+l^2)}}\mathcal{L}_0$$

where

$$\mathcal{D}_n = \partial_r + \mathrm{i}\frac{\omega}{f(r)} + 2n\frac{r-M}{\Delta}$$

$$\mathcal{D}_n^\dagger = \partial_r - \mathrm{i}\frac{\omega}{f(r)} + 2n\frac{r-M}{\Delta}$$

$$\mathcal{L}_n = \partial_\theta + Q + n\cot\theta$$

$$\mathcal{L}_n^\dagger = \partial_\theta - Q + n\cot\theta \tag{9}$$

in which $\Delta = r^2 - 2Mr - l^2$ and $Q = 2l\omega\cot\theta + m\csc\theta$. We note that $\mathcal{L}_n$ is (up to a factor $\frac{1}{\sqrt{2(r^2+l^2)}}$) nothing but one of the weighted derivative operators introduced in the compacted-spin coefficient formalism [13]. Now substituting the above operators in the Maxwell equations (6) we have;

$$\frac{1}{\sqrt{2(r^2+l^2)}}\mathcal{L}_1\phi_0 = (\mathcal{D}_0 + 2\frac{r-\mathrm{i}l}{r^2+l^2})\phi_1$$

$$\frac{1}{\sqrt{2(r^2+l^2)}}\mathcal{L}_0\phi_1 = (\mathcal{D}_0 + \frac{r}{r^2+l^2})\phi_2$$

$$\frac{1}{\sqrt{2(r^2+l^2)}}\mathcal{L}_1^\dagger\phi_2 = -\frac{f(r)}{2}(\mathcal{D}_0^\dagger + 2\frac{r-\mathrm{i}l}{r^2+l^2})\phi_1$$

$$\frac{1}{\sqrt{2(r^2+l^2)}}\mathcal{L}_0^\dagger\phi_1 = -\frac{f(r)}{2}(\mathcal{D}_1^\dagger - \frac{r}{r^2+l^2})\phi_0$$

The above equations will look more compact if we change the variables to;

$$\Phi_0 = \phi_0 \quad , \quad \Phi_1 = \sqrt{2(r^2+l^2)}\phi_1 \quad , \quad \Phi_2 = 2(r^2+l^2)\phi_2$$

where now, using the definition of the operator $\mathcal{D}_n$, we have;

$$\mathcal{L}_1\Phi_0 = (\mathcal{D}_0 + \frac{r-2\mathrm{i}l}{r^2+l^2})\Phi_1 \tag{10}$$

$$\mathcal{L}_0\Phi_1 = (\mathcal{D}_0 - \frac{r}{r^2+l^2})\Phi_2 \tag{11}$$

$$\mathcal{L}_1^\dagger\Phi_2 = -\Delta(\mathcal{D}_0^\dagger + \frac{r-2\mathrm{i}l}{r^2+l^2})\Phi_1 \tag{12}$$

$$\mathcal{L}_0^\dagger\Phi_1 = -\Delta(\mathcal{D}_1^\dagger - \frac{r}{r^2+l^2})\Phi_0 \tag{13}$$

The commutativity of the operators $\mathcal{L}_0^\dagger$ and $-\Delta(\mathcal{D}_0 + \frac{r-2\mathrm{i}l}{r^2+l^2})$ enables us to eliminate Φ_1 from equations (10) and (13) and get;

$$\left[\mathcal{L}_0^\dagger\mathcal{L}_1 + \Delta(\mathcal{D}_1 + \frac{r-2\mathrm{i}l}{r^2+l^2})(\mathcal{D}_1^\dagger - \frac{r}{r^2+l^2})\right]\Phi_0(r,\theta) = 0 \tag{14}$$

where we have used the fact that $\mathcal{D}_n\Delta = \Delta\mathcal{D}_{n+1}$. Similarly the commutativity of the operators $\mathcal{L}_0$ and $-\Delta(\mathcal{D}_0^\dagger + \frac{r-2\mathrm{i}l}{r^2+l^2})$ will allow us to eliminate Φ_1 from equations (11) and (12) and get;

$$\left[\mathcal{L}_0\mathcal{L}_1^\dagger + \Delta(\mathcal{D}_0^\dagger + \frac{r-2\mathrm{i}l}{r^2+l^2})(\mathcal{D}_0 - \frac{r}{r^2+l^2})\right]\Phi_2(r,\theta) = 0 \tag{15}$$

Now our task is to find solutions to the equations (14) and (15) by separating them into angular and radial parts. This separation can be achieved by choosing the following general form for $\Phi_0(r,\theta)$ and $\Phi_2(r,\theta)$;

$$\Phi_0(r,\theta) = \mathcal{R}_0(r)\Theta_0(\theta) \qquad \text{and} \qquad \Phi_2(r,\theta) = \mathcal{R}_2(r)\Theta_2(\theta) \tag{16}$$

Substituting (16) in (14) and (15) we have;

$$\mathcal{L}_0^{\dagger}\mathcal{L}_1\Theta_0(\theta) = (\partial_\theta - Q)(\partial_\theta + Q + \cot\theta)\Theta_0(\theta) = -\xi\Theta_0(\theta) \tag{17a}$$

$$\Delta(\mathcal{D}_1 + \frac{r - 2il}{r^2 + l^2})(\mathcal{D}_1^{\dagger} - \frac{r}{r^2 + l^2})\mathcal{R}_0(r) = \xi\mathcal{R}_0 \tag{17b}$$

$$\mathcal{L}_0\mathcal{L}_1^{\dagger}\Theta_2(\theta) = (\partial_\theta + Q)(\partial_\theta - Q + \cot\theta)\Theta_2(\theta) = -\xi\Theta_2(\theta) \tag{18a}$$

$$\Delta(\mathcal{D}_0^{\dagger} + \frac{r - 2il}{r^2 + l^2})(\mathcal{D}_0 - \frac{r}{r^2 + l^2})\mathcal{R}_2(r) = \xi\mathcal{R}_2 \tag{18b}$$

where ξ is the separation constant. These are analogous to the Teukolsky equations in Kerr space [11,14]. Note that we have not distinguished between the separation constants that derived from equations (14) and (15) and the reason is that equations (17a) and (18a) determine the same set of eigenvalues ξ. This can be seen by considering equation (17a) where, apart from the regularity requirement of $\Theta_0(\theta)$ at $\theta = 0$ and $\theta = \pi$ for the determination of ξ, the operator acting on $\Theta_2(\theta)$ in equation (18a) is the same as the operator acting on Θ_0 in (17a) if we replace θ by $\pi - \theta$ and this is the case because the general relation $\mathcal{L}_n(\theta) = -\mathcal{L}_n^{\dagger}(\pi - \theta)$ holds. So a proper solution , $\Theta_0(\theta;\xi)$, of equation (17a) with eigenvalue ξ , is also a solution of equation (18a) with the same eigenvalue, if we replace θ by $\pi - \theta$ in $\Theta_0(\theta;\xi)$.

III. SOLUTION TO THE ANGULAR EQUATION

In this section we solve equation (17a) analytically. It is obvious from the form of the equation (18a) and the discussion above that it will have the same set of eigenvalues as (17a). First of all we rewrite equation (17a) in the following form;

$$(\partial_\theta^2 + \cot\theta\partial_\theta)\Theta_0(\theta) + \Theta_0(\theta)\partial_\theta(Q + \cot\theta) - (Q^2 + Q\cot\theta - \xi)\Theta_0(\theta) = 0$$

Substituting for Q from its definition, the above equation becomes;

$$(\partial_\theta^2 + \cot\theta\partial_\theta)\Theta_0(\theta) - \frac{1}{\sin^2\theta}(2l\omega + 1)\Theta_0(\theta) - \left(\frac{2m\cos\theta}{\sin^2\theta} + 4l^2\omega^2\cot^2\theta + \frac{m^2}{\sin^2\theta} + 4l\omega m\frac{\cos\theta}{\sin^2\theta} + 2l\omega\cot^2\theta - \xi\right)\Theta_0(\theta) = 0$$

Now changing the variable to $x = \cos\theta$ we have;

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{dx}^2} - 2x\frac{\mathrm{d}}{\mathrm{dx}} - \frac{(2mx + 4b^2x^2 + m^2 + 4bmx + 2bx^2 + 2b + 1)}{1-x^2} + \xi\right]\Theta_0(x) = 0 \quad (19)$$

where $b = l\omega$. Dominant behaviour of the eigenfunctions of the above equation at singular points $x = \pm 1$ are given by $(1 \mp x)^{\frac{|1+2b\pm m|}{2}}$ respectively. Introducing $\alpha = |1 + 2b + m|$ and $\beta = |1 + 2b - m|$, and substituting for Θ_0 the following expression;

$$(\frac{1-x}{2})^{\alpha/2}(\frac{1+x}{2})^{\beta/2}U(x)$$

in (19) it will be transformed into;

$$(1-x^2)\frac{\mathrm{d}^2U(x)}{\mathrm{dx}^2} + [\beta - \alpha - (2 + \alpha + \beta)x]\frac{\mathrm{d}U(x)}{\mathrm{dx}} + [\xi - (\frac{\alpha+\beta}{2})(\frac{\alpha+\beta}{2} + 1) + 4b^2 + 2b]U(x) = 0 \quad (20a)$$

This is the differential equation satisfied by the Jacobi polynomials [15,16]

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$

if we identify;

$$\xi + 4b^2 + 2b = [n + (\frac{\alpha+\beta}{2})][n + (\frac{\alpha+\beta}{2} + 1)] = j(j+1) \quad (20b)$$

where $j = n + (\frac{\alpha+\beta}{2})$. So Θ_0 could be written explicitly as follows

$$\Theta_0(x) = \frac{(-1)^n}{2^{n-\frac{\alpha}{2}-\frac{\beta}{2}}n!}(1-x)^{-\frac{\alpha}{2}}(1+x)^{-\frac{\beta}{2}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \quad (20c)$$

IV. SOLUTIONS TO THE RADIAL EQUATIONS

Now we try to solve the radial equations (17b) and (18b) by replacing for the operators $\mathcal{D}$ and $\mathcal{D}_1$ and their complex conjugates from (9);

$$\partial_r^2\mathcal{R}_0 + (\frac{4[r-M]}{\Delta} - 2\frac{\mathrm{i}l}{r^2+l^2})\partial_r\mathcal{R}_0 + \left(\frac{\omega^2(r^2+l^2)^2 - 2\mathrm{i}\omega M(l^2-r^2) + 4\mathrm{i}r\omega l^2}{\Delta^2}\right)\mathcal{R}_0$$

$$+\left(\frac{2 - 2\mathrm{i}\omega(r - \mathrm{i}l) - \xi}{\Delta} - \frac{4\mathrm{i}l(r-M)}{\Delta(r^2+l^2)} + \frac{2\mathrm{i}lr - l^2}{(r^2+l^2)^2}\right)\mathcal{R}_0 = 0 \quad (21a)$$

$$\partial_r^2 \mathcal{R}_2 - 2\frac{\mathrm{i}l}{r^2+l^2}\partial_r \mathcal{R}_2$$

$$+\left(\frac{2\mathrm{i}\omega M(l^2-r^2) - 4\mathrm{i}r\omega l^2 + \omega^2(r^2+l^2)^2}{\Delta^2} + \frac{2\mathrm{i}\omega(r-\mathrm{i}l)-\xi}{\Delta} + \frac{2\mathrm{i}lr - l^2}{(r^2+l^2)^2}\right)\mathcal{R}_2 = 0 \qquad (21b)$$

where $\Delta = r^2 - 2Mr - l^2 = (r-r_+)(r-r_-)$ with $r\pm = M \pm \sqrt{M^2+l^2}$ and r_+ being the position of the horizon. Therefore in our discussion below we always have $r \geq r_+$.

A. Solution to the equation (21b)

We start by solving equation (21b) which looks simpler. This equation has four regular singularities at $r = r_\pm$ and $r = \pm\mathrm{i}l$ and an irregular singularity at $r = \infty$. To remove the singularities at $r = \pm\mathrm{i}l$ we use the following transformation ;

$$\mathcal{R}_2(r) = (\frac{r-\mathrm{i}l}{r+\mathrm{i}l})^{1/2}\mathcal{V}_2(r) \qquad (22a)$$

so that equation (21b) becomes (after some manipulation);

$$\partial_r^2 \mathcal{V}_2(r) + \left(\frac{A}{(r-r_+)^2} + \frac{B}{(r-r_-)^2} + \frac{C}{(r-r_+)(r-r_-)} + D(\frac{1}{r-r_+} + \frac{1}{r-r_-}) + \omega^2\right)\mathcal{V}_2(r) = 0 \qquad (22b)$$

where

$$A = r_+(2M\omega^2 - \mathrm{i}\omega) + b^2$$

$$B = r_-(2M\omega^2 - \mathrm{i}\omega) + b^2$$

$$C = 2M(2M\omega^2 + \mathrm{i}\omega) + 2(b^2+b) - \xi$$

$$D = 2M\omega^2 + \mathrm{i}\omega \qquad (23)$$

Now if we define

$$r - r_+ = -\varepsilon x \qquad \& \qquad r - r_- = \varepsilon(1-x) \qquad (24)$$

where $\varepsilon = 2(M^2+l^2)^{1/2}$ and take

$$\mathcal{V}_2 = (-x)^\alpha (1-x)^\beta \bar{\mathcal{V}}_2$$

(where α and β are constants to be determined later) then equation (22b) will transform into;

$$\partial_x^2\bar{\mathcal{V}}_2(x)+2(\frac{\alpha}{x}-\frac{\beta}{1-x})\partial_x\bar{\mathcal{V}}_2(x)+\left[\frac{A}{x^2}+\frac{B}{(1-x)^2}-\frac{C}{x(1-x)}\right]\bar{\mathcal{V}}_2(x)$$

$$+\left[\varepsilon D(\frac{1}{1-x}-\frac{1}{x})+\varepsilon^2\omega^2+\frac{\alpha(\alpha-1)}{x^2}-\frac{2\alpha\beta}{x(1-x)}+\frac{\beta(\beta-1)}{(1-x)^2}\right]\bar{\mathcal{V}}_2(x)=0 \tag{25}$$

Now in order to eliminate the terms proportional to $\frac{1}{x^2}$ and $\frac{1}{(1-x)^2}$, α and β should take one of the following values. ;

$$\alpha_\pm=\frac{1}{2}(1\pm\sqrt{1-4A}) \qquad , \qquad \beta_\pm=\frac{1}{2}(1\pm\sqrt{1-4B}) \tag{26}$$

As in the case of the Kerr spacetime one can show that different choices of the above parameters correspond to solutions which satisfy the incoming and outgoing boundary conditions [17]. For example choosing the pair (α_-,β_+) corresponds to the solution satisfying the incoming boundary condition[2]. With the above choice of α and β equation (25) reduces to;

$$\partial_x^2\bar{\mathcal{V}}_2(x)+2(\frac{\alpha}{x}-\frac{\beta}{1-x})\partial_x\bar{\mathcal{V}}_2(x)+\left[\varepsilon D(\frac{1}{1-x}-\frac{1}{x})\right]\bar{\mathcal{V}}_2(x)-$$
$$\left[\frac{C}{x(1-x)}+\varepsilon^2\omega^2-\frac{2\alpha\beta}{x(1-x)}\right]\bar{\mathcal{V}}_2(x)=0 \tag{27}$$

Now choosing $\bar{\mathcal{V}}_2(x)=\exp(i\varepsilon\omega x)\tilde{\mathcal{V}}_1(x)$ the above equation can be written in the following form;

$$x(1-x)\partial_x^2\tilde{\mathcal{V}}_2(x)+[2\alpha-2(\alpha+\beta)x]\partial_x\tilde{\mathcal{V}}_2(x)-pq\tilde{\mathcal{V}}_2(x)=$$
$$((2M+\varepsilon)(2M\omega^2+\mathrm{i}\omega)+2(3l^2\omega^2+2l\omega)-j(j+1)+2\alpha\beta-2\mathrm{i}\varepsilon\omega\alpha)\tilde{\mathcal{V}}_2(x)+$$
$$(2\mathrm{i}\varepsilon\omega[\alpha+\beta]-2\varepsilon(2M\omega^2+\mathrm{i}\omega))x\tilde{\mathcal{V}}_2(x)-2\mathrm{i}\varepsilon\omega x(1-x)\partial_x\tilde{\mathcal{V}}_2(x)-pq\tilde{\mathcal{V}}_2(x) \tag{28a}$$

where we have substituted for C, D and ξ from equations (20a) and (23) and added $-pq\tilde{\mathcal{V}}_2(x)$ on both sides to make the left hand side of the equation to look like the hypergeometric

[2]This could also be seen by the fact that $a=0$ limit of the Kerr case and $l=0$ limit of NUT case should both reduce to the electromagnetic perturbation of schwarzschild spacetime.

equation with the identification

$$p + q = 2(\alpha + \beta) - 1 \tag{28b}$$

The right hand side of the above equation includes terms of different order in ω and therefore it is suitable for finding the solution in the expansion of this parameter and it is obvious that the zero-th order solutions of the equation are the hypergeometric series provided we choose

$$pq = 2\alpha\beta - j(j+1) \tag{28c}$$

To write the zero-th order solution more explicitly we note that for 2α a non-integer the general solution in terms of hypergeometric series could be written as follows;

$$\bar{\mathcal{V}}_2(x) = AF(p, q; 2\alpha; x) + Bx^{1-2\alpha}F(p - 2\alpha + 1, q - 2\alpha + 1; 2 - 2\alpha; x)$$

hence to zeroth order in ω we have;

$$\begin{aligned}&\mathcal{R}_2(r) \approx (\frac{r - il}{r + il})^{\frac{1}{2}}(\frac{r - r_+}{\varepsilon})^{\alpha}(\frac{r - r_-}{\varepsilon})^{\beta}\\ &\left(AF(p, q; 2\alpha; \frac{r_+ - r}{\varepsilon}) + B(\frac{r_+ - r}{\varepsilon})^{1-2\alpha}F(p - 2\alpha + 1, q - 2\alpha + 1; 2 - 2\alpha; \frac{r_+ - r}{\varepsilon})\right)\end{aligned} \tag{28d}$$

where A and B are constants and p and q are solutions of equations (28b) and (28c).

B. Solution to the equation (21a)

Amazingly enough one can see that the same procedure used to transform equation (21b) to (28) can also be applied to equation (21a). The reason for this is the fact that the same transformation as in (22a) will remove the singularities of (21a) at $r = \pm il$, transforming it into;

$$\begin{aligned}&\partial_r^2\mathcal{V}_0(r) + 2(\frac{1}{r - r_+} + \frac{1}{r - r_-})\partial_r\mathcal{V}_0(r) +\\ &\left(\frac{A'}{(r - r_+)^2} + \frac{B'}{(r - r_-)^2} + \frac{C'}{(r - r_+)(r - r_-)} + D'(\frac{1}{r - r_+} + \frac{1}{r - r_-}) + \omega^2\right)\mathcal{V}_0(r) = 0\end{aligned} \tag{29}$$

where

$$\mathcal{R}_0(r) = (\frac{r - \mathrm{i}l}{r + \mathrm{i}l})^{1/2}\mathcal{V}_0(r) \tag{30}$$

and

$$A' = r_+(2M\omega^2 + \mathrm{i}\omega) + b^2$$

$$B' = r_-(2M\omega^2 + \mathrm{i}\omega) + b^2$$

$$C' = 2M(2M\omega^2 - \mathrm{i}\omega) + 2(b^2 + b) - \xi'$$

$$D' = 2M\omega^2 - \mathrm{i}\omega \tag{31}$$

It is notable that these equations are the same as in (23) with $\mathrm{i}\omega$ transformed into $-\mathrm{i}\omega$ and ξ to $\xi' = \xi - 2$. Now applying the same change of the variable as in (24) and substituting for

$$\mathcal{V}_0(r) = (-x)^{\alpha'+1}(1-x)^{\beta'+1}\exp(i\varepsilon\omega x)\tilde{\mathcal{V}}_0(x) \tag{32a}$$

where

$$\alpha'_{\pm} = \frac{1}{2}(-3 \pm \sqrt{1 - 4A'}) \quad , \quad \beta'_{\pm} = \frac{1}{2}(-3 \pm \sqrt{1 - 4B'}) \tag{32b}$$

equation (29) will transform into

$$x(1-x)\partial_x^2\tilde{\mathcal{V}}_0(x) + [2\alpha' - 2(\alpha' + \beta')x]\partial_x\tilde{\mathcal{V}}_0(x) - p'q'\tilde{\mathcal{V}}_0(x) =$$

$$[2\mathrm{i}\varepsilon\omega(\alpha' + \beta') + 2\varepsilon(2M\omega^2 - \mathrm{i}\omega)]x\tilde{\mathcal{V}}_0(x) + 2\mathrm{i}\varepsilon\omega x(1-x)\partial_x\tilde{\mathcal{V}}_0(x) - p'q'\tilde{\mathcal{V}}_0(x) +$$

$$\Big[(2M + \varepsilon)(2M\omega^2 - \mathrm{i}\omega) + 2(3l^2\omega^2 + 2l\omega) - j'(j'+1) + 2(\alpha'+1)(\beta'+1) + 2\mathrm{i}\varepsilon\omega\alpha'\Big]\tilde{\mathcal{V}}_0(x) \tag{33}$$

where

$$j'(j'+1) = \xi' + 4b^2 + 2b \tag{34}$$

As one can see the left hand side of the above equation is also the hypergeometric equation with the identification $p' + q' = 2(\alpha' + \beta') - 1$. As in equation (28a) the right hand side of this equation includes terms of different order in ω and it is therefore suitable for finding the solution in the expansion of this parameter. Again the zero-th order solution is the hypergeometric function provided we choose

$$p'q' = j'(j'+1) - 2(\alpha'+1)(\beta'+1).$$

V. EQUATION SATISFIED BY Φ_1 AND ITS SOLUTION

To complete our discussion we need to solve the equation satisfied by scalar ϕ_1 which, using equations (11) and (12), can be written in the following form;

$$\left[\mathcal{L}_1^\dagger \mathcal{L}_0 + \Delta(\mathcal{D}_1 - \frac{r}{r^2+l^2})(\mathcal{D}_0^\dagger + \frac{r-2\mathrm{i}l}{r^2+l^2})\right]\Phi_1(r,\theta) = 0 \tag{35}$$

Now upon choosing the form

$$\Phi_1(r,\theta) = \mathcal{R}_1(r)\Theta_1$$

will transform into the following angular and radial equations;

$$\mathcal{L}_1^\dagger \mathcal{L}_0 \Theta_1(\theta) = (\partial_\theta - Q + \cot\theta)(\partial_\theta + Q)\Theta_1(\theta) = -\eta\Theta_1(\theta) \tag{36}$$

$$\Delta(\mathcal{D}_1 - \frac{r}{r^2+l^2})(\mathcal{D}_0^\dagger + \frac{r-2\mathrm{i}l}{r^2+l^2})\mathcal{R}_1(r) = \eta\mathcal{R}_1 \tag{37}$$

Now our task is to solve the above two equations.

A. solution to the angular equation

We start with (36) which can be solved in the same way as we solved equations (17a) and (18a). Substituting for Q from its definition and changing the variable to $x = \cos\theta$, equation (36) will transform into;

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{dx}^2} - 2x\frac{\mathrm{d}}{\mathrm{dx}} - \frac{(2mx + 4b^2x^2 + m^2 + 4bmx + 2bx^2 + 2b)}{1-x^2} + \eta\right]\Theta_1(x) = 0 \tag{38}$$

where $b = l\omega$. One can see that the only difference between this equation and equation (19) is the disappearance of 1 in the numerator of $1-x^2$. In a similar way to equation (19), the dominant behavior of the eigenfunctions of the above equation at singular points $x = \pm 1$ are given by $(1 \mp x)^{\frac{\sqrt{\mathcal{B}_\pm^2 - 1}}{2}}$ where $\mathcal{B}_\pm = |1 + 2b \pm m|$. Introducing $\alpha' = (\mathcal{B}_+ + 1)^{1/2}(\mathcal{B}_+ - 1)^{1/2}$ and $\beta' = (\mathcal{B}_- + 1)^{1/2}(\mathcal{B}_- - 1)^{1/2}$, and substituting for Θ_1 the following expression;

$$(\frac{1-x}{2})^{\alpha'/2}(\frac{1+x}{2})^{\beta'/2}U'(x)$$

in (38) it will be transformed into;

$$(1-x^2)\frac{\mathrm{d}^2U'(x)}{\mathrm{dx}^2}+[\beta'-\alpha'-(2+\alpha'+\beta')x]\frac{\mathrm{d}U'(x)}{\mathrm{dx}}+[\eta-(\frac{\alpha'+\beta'}{2})(\frac{\alpha'+\beta'}{2}+1)+4b^2+2b]U'(x)=0 \tag{39}$$

This is again the differential equation satisfied by the Jacobi polynomials if we identify;

$$\eta+4b^2+2b=[n+(\frac{\alpha'+\beta'}{2})][n+(\frac{\alpha'+\beta'}{2}+1)]=j(j+1) \tag{40}$$

where $j=n+(\frac{\alpha'+\beta'}{2})$.

B. solution to the radial equation

Substituting for the operators from their definitions given in (9), the radial equation (37) can be written in the following form;

$$\partial_r^2\mathcal{R}_1(r)+2(\frac{r-M}{\Delta}-\frac{\mathrm{i}l}{r^2+l^2})\partial_r\mathcal{R}_1(r)+\left(\frac{\omega^2(r^2+l^2)^2}{\Delta^2}\right)\mathcal{R}_1(r)$$

$$-\left(\frac{2\omega l+\eta}{\Delta}-\frac{2(r-M)(r-2\mathrm{i}l)}{\Delta(r^2+l^2)}-\frac{6\mathrm{i}lr+l^2-2r^2}{(r^2+l^2)^2}\right)\mathcal{R}_1(r)=0 \tag{41}$$

One can remove the singularity at $r=il$ by the following transformation;

$$\mathcal{R}_1(r)=\frac{(r-il)^{1/2}}{(r+il)^{3/2}}\mathcal{V}_1(r) \tag{42}$$

so that equation (41) transforms into;

$$\partial_r^2\mathcal{V}_1+2(\frac{r-M}{\Delta}-\frac{1}{r+il})\partial_r\mathcal{V}_1+\left(\frac{\omega^2(r^2+l^2)^2}{\Delta^2}-\frac{2\omega l+\eta}{\Delta}\right)\mathcal{V}_1=0 \tag{43}$$

Then by the following change of variable,

$$\frac{r-r_+}{2M}=-z$$

the above equation transforms into

$$\partial_z^2\mathcal{V}_1+\left(\frac{1}{z}+\frac{1}{z-1}-\frac{2}{z-a}\right)\partial_z\mathcal{V}_1+\frac{\alpha\beta z-q}{z(z-1)(z-a)}\mathcal{V}_1=\frac{\omega^2}{2M^3}\frac{[(r_+-2Mz)^2+l^2]^2}{z^2(z-1)^2}\mathcal{V}_1 \quad (44a)$$

where

$$\alpha + \beta = -1 \ \ ; \ \ \alpha\beta = \frac{1}{2M}(2\omega l + \eta) \tag{44b}$$

and

$$q = \alpha\beta a \ \ ; \ \ a = \frac{r_+ + \mathrm{i}l}{2M} \tag{44c}$$

Comparing equations (44a-c) with Heun's differential equation [17] defined by

$$\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0 \tag{45}$$

where

$$a \neq 0, 1 \ \text{ and } \ \gamma + \delta + \epsilon = \alpha + \beta + 1$$

we find out that equation (44a), along with the fact that $|a| > 1$, forms the Heun equation with the right hand side including a term of second order in ω, allowing solutions in the expansion of this parameter. Again the zeroth order solutions are solutions to the Heun equation but before looking for that the following two points worth to be mentioned;

First we note that despite the fact that $q = a\alpha\beta$, the left hand side of equation (44a) does not degenerate into hypergeometric equation because $\epsilon \neq 0$ and secondly, due to the fact that all the parameters γ, δ and ϵ are integers we expect that the general solution to the Heun equation involve logarithmic terms.

To write the explicit zero-th order solution to the equation (44a) we only consider the basic power series solution [3] to the Heun equation denoted by [18];

$$Hl(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{j=0}^{\infty} c_j z^j \quad , \quad (c_0 = 1) \tag{46}$$

where Hl stands for 'Heun local' solution and apart from the normalized coefficint c_0 the other coefficints are given by the following three-term recursion relation;

$$-qc_0 + a\gamma c_1 = 0 \tag{47}$$

[3]For detailed dicussion on possible different solutions to Heun's equation see [18-19]

$$P_j c_{j-1} - (Q_j + q)c_j + R_j c_{j+1} = 0 \ \ , \ \ (j \geq 1) \tag{48}$$

in which ;

$$P_j = (j - 1 + \alpha)(j - 2 - \alpha) \tag{49a}$$

$$Q_j = j[j(1 + a) + a - 2] \tag{49b}$$

$$R_j = a(j + 1)^2 \tag{49c}$$

where we used relations (44b) and the fact that in our case, left hand side of equation (44a),

$$\gamma = \delta = 1 \quad \text{and} \quad \epsilon = -2$$

Therefore the zero-th order solution to the equation (44a) is given by;

$$\mathcal{R}_1(r) = \frac{(r - il)^{1/2}}{(r + il)^{3/2}} Hl(a, q; \alpha, \beta, 1, 1; \frac{r_+ - r}{2M}) \tag{50}$$

It is also interesting to note that Heun's equation and its biconfluent extension also appear in the discussions on the electromagnetic perturbations of Kerr-de Sitter and Kasner spacetimes respectively and indeed furnish exact solutions to the Maxwell equations in these spacetimes [20,21].

VI. RESULTS AND DISCUSSION

In this paper we have considered the Maxwell equations in NUT space and solved them analytically using the Newman-Penrose null tetrad formalism, which is best adapted for treating type-D spacetimes. It is shown that after separation of the equations, solutions to the angular equations could be given in terms of the Jacobi polynomials. The radial equations on the other hand are transformed into hypergeometric and Heun's equations which on the right hand side have terms of different order in ω. This fact enables one to find solutions in the expansion of this parameter with the zero-th order solutions being the hypergeometric function and possible solutions to the Heun equation respectively. To compare the results with the Schwarzscild case we note that the appearance of the perturbation frequency ω in

the angular equations (19) and (38) , only through the parameter $b = l\omega$, means that unlike Schwarzschild, in the stability discussions of NUT space, we can not only consider radial equations and angular equations should also be considered. In this respect the behaviour of NUT space under electromagnetic pertubations is very similar to that of Kerr in which angular equations depend on the frequency through the combination $a\omega$ [11] . So in both cases going over to the Schwarzschild space will remove the perturbation parameter from the angular equation and the stability discussion will only need a study of radial equations. Another point deserving attention is to check whether, as in the case of the Kerr geometry, it is possible to write the solutions to the radial equations in the form of an expansion in terms of the coulomb wave functions [17,22].

ACKNOWLEDGEMENTS

I would like to thank D. Lynden-Bell for his valuable comments on the first draft of the paper and J. Bicak for useful discussions. The author thanks University of Tehran for supporting this project under the grants provided by the Research Council.

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